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Complex Maxwell's equations

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A unified complex model of Maxwell's equations is presented. The wave nature of the electromagnetic field vector is related to the temporal and spatial distributions and the circulation of charge and current densities. A new vacuum solution is obtained, and a new transformation under which Maxwell's equations are invariant is proposed. This transformation extends ordinary gauge transformation to include charge-current as well as scalar-vector potential. An electric dipole moment is found to be related to the magnetic charges, and Dirac's quantization is found to determine an uncertainty relation expressing the indeterminacy of electric and magnetic charges. We generalize Maxwell's equations to include longitudinal waves. A formal analogy between this formulation and Dirac's equation is also discussed.

Keywords: Maxwell's equations, duality transformations, magnetic charge (monopole)

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1. Introduction

Maxwell's theory of electromagnetism is one of the most successful theories that nature has endowed.[1] Maxwell's equations were first expressed by a quaternion involving 20 quantities. Although Maxwell wrote his equation into a quaternionic form, he never made use of this formalism directly, instead he used the components of the quaternionic quantities. Quaternions were invented by Hamilton in 1843. When vectors were invented by Gibbs and Heaviside, Maxwell's equations (in their present form) were a set of four equations: two scalar equations and two vector equations. We have recently used a quaternion and formulated Maxwell's equation in a single quaternionic equation.[2] In tensor form, Maxwell's equations are two tensorial equations. We now redefine the electric and magnetic fields by a complex vector (electromagnetic vector $\boldsymbol{F}$). This vector is spanned by the electric and magnetic field vectors, $\boldsymbol{E}$ and $\boldsymbol{B}$. With this, Maxwell's equations become two equations: one vector equation and one scalar equation.[3–6] We first introduce the electromagnetic complex vector, and rewrite Maxwell's equations in terms of this vector. We then write the wave equation arising from this vector, as well as the energy conservation equation, and show that the electromagnetic wave travels at the speed of light not only in vacuum, but also in any medium satisfying the generalized continuity equation. Such media are endowed with a steady current density with no circulation, and spatially uniform charge distribution. These properties need the vorticity arising from the flow of the particles in the medium to be tantamount to the magnetic field generated by a moving charge in an electric field. We further show that the present form of Maxwell's equations preserves the duality transformation if we extend our transformation to include space and time, and we compare the new Maxwell's equations with Dirac's equation. An analogy is drawn from this formulation, and some new interpretation of the electromagnetic complex vector is given, along with a symmetric formulation of Maxwell's equations. It is found that in free space, the electromagnetic vector precesses with an angular velocity whose corresponding frequency is equal to the frequency of the emitted electromagnetic wave.

Owing to the analogy between the Lorenz gauge condition and charge conservation, we suggest an extended gauge transformation to include charge and current densities. Proca equations[7] that extend Maxwell's equations to include massive photons are then invariant under this extended gauge transformation. We finally show that the duality transformation results from rotating the electric and magnetic fields, where the resulting force is the dual transformation of Lorentz force in an inertial frame. We show that the generalized Lorentz force associated with monopoles stems from the ordinary Maxwell's equations. Maxwell equations are then generalized to include longitudinal waves. These generalized equations are shown to be invariant under the duality transformations.

2. The complex electromagnetic vector

Now we define the electromagnetic fields in terms of the complex vector as[5]

$$\boldsymbol{F} = \frac{\boldsymbol{E}}{c} + \mathrm{i}\boldsymbol{B}, \tag{1}$$

where c is the speed of light in vacuum. With this definition, Maxwell's equations with the source having charge (ρ) and current ($\boldsymbol{J}$) densities can be written as

$$\frac{\partial \boldsymbol{F}}{\partial t} + \mathrm{i}c\boldsymbol{\nabla}\times\boldsymbol{F} = -\mu_0 c\boldsymbol{J}, \tag{2}$$

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and

$$\nabla\cdot\boldsymbol{F}=c\mu_0\rho, \tag{3}$$

where ε_0 and μ_0 are the permittivity and permeability of free space, respectively, and $c^{-2}=\sqrt{\varepsilon_0\mu_0}$. It is interesting to know that in the Whittaker representation,[6] the electromagnetic vector enables us to accommodate full solutions of the Maxwell equations into just one complex function. Moreover, Bialynicki–Birula and Bialynicka–Birula investigated the angular momentum carried by a beam of electromagnetic radiation using analogous formulation.[7] Equation (2) can be compared with the derivative of a vector in a rotating frame and compared to its derivative in a fixed frame, namely,

$$\left(\frac{\partial\boldsymbol{A}}{\partial t}\right)_f=\left(\frac{\partial\boldsymbol{A}}{\partial t}\right)_r+\boldsymbol{\omega}\times\boldsymbol{A}. \tag{4}$$

Since for a plane wave $\mathrm{i}c\nabla\times\boldsymbol{F}=\boldsymbol{\omega}\times\boldsymbol{F}$, where $\boldsymbol{\omega}=c\boldsymbol{k}$ represents the angular velocity of the rotating frame (coordinates), $\boldsymbol{B}$ behaves like a small scale rotating frame with respect to $\boldsymbol{E}$, and the comparison is plausible. Hence, one can write Eq. (2) in a simple form

$$D_\mathrm{t}\boldsymbol{F}=-\mu_0c\boldsymbol{J},\quad D_\mathrm{t}=\frac{\partial}{\partial t}+\mathrm{i}c\nabla\times, \tag{5}$$

where D_t is the derivative operator as seen by an observer in a fixed frame. Therefore, Maxwell's equations in a fixed (inertial) frame take the simple forms, i.e., Eqs. (4) and (5).

If we now use the vector identity,

$$\begin{aligned}\nabla\times(\boldsymbol{v}\times\boldsymbol{F}) &= \boldsymbol{v}(\nabla\cdot\boldsymbol{F})-\boldsymbol{F}(\nabla\cdot\boldsymbol{v})\\ &\quad+(\boldsymbol{F}\cdot\nabla)\boldsymbol{v}-(\boldsymbol{v}\cdot\nabla)\boldsymbol{F}.\end{aligned} \tag{6}$$

For a constant velocity ($\boldsymbol{v}$), one finds

$$\nabla\cdot\boldsymbol{v}=0,\quad(\boldsymbol{F}\cdot\nabla)\boldsymbol{v}=0, \tag{7}$$

so that

$$\nabla\times(\boldsymbol{v}\times\boldsymbol{F})=\boldsymbol{v}(\nabla\cdot\boldsymbol{F})-(\boldsymbol{v}\cdot\nabla)\boldsymbol{F}. \tag{8}$$

Note that

$$\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}t}=\frac{\partial\boldsymbol{F}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{F}, \tag{9}$$

so that upon using Eqs. (2), (3) and (8), one finds

$$q\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}t}=\nabla\times\boldsymbol{f}, \tag{10}$$

where

$$\boldsymbol{f}=-\mathrm{i}qc\boldsymbol{F}-q\boldsymbol{v}\times\boldsymbol{F}, \tag{11}$$

and we have used the relation $\boldsymbol{J}=\rho\boldsymbol{v}$, and q is the charge of the particle. The term $q\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}t}$ in Eq. (10) can be interpreted as the work per unit area performed by the electromagnetic fields on the charged particle. It is remarkable to see that Maxwell's equations, in particular Eq. (10), embody the generalized Lorentz force, $\boldsymbol{f}$, which gives the force on a particle with electric and magnetic charges. This issue is dealt with in Section 5, where this force appears naturally as a result of considering the symmetric Maxwell's equations.

Inserting Eq. (1) into Eqs. (2) and (3) and equating the real and imaginary parts on both sides for each of these equations, we obtain

$$\nabla\cdot\boldsymbol{E}=\frac{\rho}{\varepsilon_0},\quad\nabla\cdot\boldsymbol{B}=0. \tag{12}$$

and

$$\nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t},\quad\nabla\times\boldsymbol{B}=\mu_0\boldsymbol{J}+\frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t}. \tag{13}$$

The solution of Maxwell's equations, Eqs. (2) and (3), can be obtained by taking the curl of Eq. (2) and employing the gradient of Eq. (3) and the vector identity

$$\nabla\times(\nabla\times\boldsymbol{F})=\nabla(\nabla\cdot\boldsymbol{F})-\nabla^2\boldsymbol{F}. \tag{14}$$

We thus obtain

$$\frac{1}{c^2}\frac{\partial^2\boldsymbol{F}}{\partial t^2}-\nabla^2\boldsymbol{F}=-c\mu_0\left(\nabla\rho+\frac{1}{c^2}\frac{\partial\boldsymbol{J}}{\partial t}\right)+\mathrm{i}\mu_0\nabla\times\boldsymbol{J}. \tag{15}$$

We can alternatively use the Arbab-Yassein commutator brackets [9, 10] and $[\partial/\partial t,\nabla]\times\boldsymbol{F}=0$ to obtain Eq. (15). Therefore, the solution of Maxwell's equations, Eq. (15), is equivalent to $[\partial/\partial t,\nabla]\times\boldsymbol{F}=0$.

If we define the complex quantity

$$\boldsymbol{Q}=-c\mu_0\left(\nabla\rho+\frac{1}{c^2}\frac{\partial\boldsymbol{J}}{\partial t}\right)+\mathrm{i}\mu_0\nabla\times\boldsymbol{J}, \tag{16}$$

then equation (15) becomes

$$\frac{1}{c^2}\frac{\partial^2\boldsymbol{F}}{\partial t^2}-\nabla^2\boldsymbol{F}=\boldsymbol{Q}. \tag{17}$$

Using Eq. (1) and equating the real and imaginary parts on both sides of Eq. (15), one can obtain

$$\frac{1}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2}-\nabla^2\boldsymbol{E}=-c^2\mu_0\left(\nabla\rho+\frac{1}{c^2}\frac{\partial\boldsymbol{J}}{\partial t}\right), \tag{18}$$

$$\frac{1}{c^2}\frac{\partial^2\boldsymbol{B}}{\partial t^2}-\nabla^2\boldsymbol{B}=\mu_0\nabla\times\boldsymbol{J}. \tag{19}$$

Taking the divergence of Eq. (2) and using Eq. (3) yields the continuity equation (charge conservation)

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\boldsymbol{J}=0. \tag{20}$$

We can alternatively obtain Eq. (20) if we employ the Arbab-Yassein commutator brackets [9, 10] and $[\partial/\partial t,\nabla]\cdot\boldsymbol{F}=0$, as well as Eqs. (2) and (3). Hence, the continuity equation is equivalent to the commutator bracket, $[\partial/\partial t,\nabla]\cdot\boldsymbol{F}=0$.

3. The new vacuum solution

Equation (17) demonstrates that $\boldsymbol{F}$ satisfies a wave equation with a vector source term, $\boldsymbol{Q}$. However, in a free space (or vacuum) where $\boldsymbol{J}=0$ and $\rho=0$ ($\boldsymbol{Q}=0$), the electromagnetic field $\boldsymbol{F}$ in Eq. (17) travels at the speed of light. It is interesting to note that there is another solution where $\boldsymbol{F}$ travels at the speed of light. Thus, the vacuum is not necessarily empty (void) when $\boldsymbol{J}\neq0$ and $\rho\neq0$, and governed by[11]

$$\nabla\rho+\frac{1}{c^2}\frac{\partial\boldsymbol{J}}{\partial t}=0,\quad\nabla\times\boldsymbol{J}=0, \tag{21}$$

in addition to Eq. (20). The solution of Eqs. (20) and (21) implies that

$$\frac{1}{c^2}\frac{\partial^2 \boldsymbol{J}}{\partial t^2} - \boldsymbol{\nabla}^2 \boldsymbol{J} = 0, \quad \frac{1}{c^2}\frac{\partial^2 \rho}{\partial t^2} - \boldsymbol{\nabla}^2 \rho = 0. \tag{22}$$

Equations (20) and (21) are invariant under the transformation

$$\boldsymbol{J}' = \boldsymbol{J} + \boldsymbol{\nabla}\lambda, \quad \rho' = \rho - \frac{1}{c^2}\frac{\partial \lambda}{\partial t}, \tag{23}$$

if the scalar function λ satisfies the wave equation

$$\frac{1}{c^2}\frac{\partial^2 \lambda}{\partial t^2} - \boldsymbol{\nabla}^2 \lambda = 0. \tag{24}$$

This reminds us of the gauge transformation of the vector potential $\boldsymbol{A}$ and scalar potential φ, under which the electric and magnetic fields are invariant. Furthermore, it is shown by Aharonov and Bohm that $\boldsymbol{A}$ and φ are physically observable quantities and not merely mathematical functions.[12] The current-field analogy can be seen if we look at the similarity between the Lorenz gauge condition and the continuity equation. This urges us to think of the continuity as a gauge condition. Hence, both equations should undergo some gauge-like transformation. In this sense one can also enlarge the gauge transformation to include $\boldsymbol{J}$ and ρ, and under this transformation Maxwell's equations (15) will be invariant. Though Maxwell's equations include both the fields and their sources (ρ and $\boldsymbol{J}$), gauge transformation is carried only to fields but not to the sources. Because of the analogy existing between the two, one should subject Maxwell's equation to both entities. Furthermore, the massive photon field introduced by Proca,[7] that extends Maxwell's equations, is found to be invariant under the enlarged gauge transformations. However, Proca equations are not invariant under the ordinary gauge transformation despite their Lorentz invariance. If we adopt the transformation (23) in addition to the ordinary gauge transformation, Proca equations can become invariant by properly choosing φ & $\boldsymbol{A}$ and ρ & $\boldsymbol{J}$.

If we consider a conducting medium satisfying this system of equations, (20) and (21), which are called the generalized continuity equations, then Ohm's law ($\boldsymbol{J} = \sigma \boldsymbol{E}$) in Eqs. (20) and (21) implies that

$$\frac{\partial \boldsymbol{E}}{\partial t} = -D\boldsymbol{\nabla}^2 \boldsymbol{E}, \quad \boldsymbol{\nabla} \times \boldsymbol{E} = 0, \quad D = \frac{1}{\mu_0 \sigma}. \tag{25}$$

This is a magnetostatic system where the electric field diffuses with a negative diffusion constant. If this condition occurs in ordinary metals, then the value of D is in the range 0.01 m^2/s–0.10 m^2/s. If we now consider a fluid described by the current density $\boldsymbol{J} = \rho \boldsymbol{v}$, then equations (20) and (21) define the vorticity of the fluid as

$$\boldsymbol{\omega}_v = \boldsymbol{\nabla} \times \boldsymbol{v} = \frac{\boldsymbol{v}}{c^2} \times \left(\frac{-\partial \boldsymbol{v}}{\partial t}\right). \tag{26}$$

This can be compared with the magnetic field due to a charged particle moving with constant velocity in an electric field, which is given by Biot–Savart law as

$$\boldsymbol{B} = \frac{\boldsymbol{v}}{c^2} \times \boldsymbol{E}. \tag{27}$$

Equations (26) and (27) suggest that for the fluid

$$\boldsymbol{a} = \boldsymbol{E}_{\rm f} = -\frac{\partial \boldsymbol{v}}{\partial t}, \tag{28}$$

which represents the fluid acceleration. For a particle of mass m, its equation of motion is

$$\boldsymbol{f} = m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t},$$

so that

$$\boldsymbol{\nabla} \times \boldsymbol{f} = m\frac{\mathrm{d}\boldsymbol{\nabla} \times \boldsymbol{v}}{\mathrm{d}t} = m\frac{\mathrm{d}\boldsymbol{\omega}_v}{\mathrm{d}t}.$$

Comparing the above equation with Eq. (10) yields

$$\boldsymbol{\omega}_{\rm v} = \frac{q}{m}\boldsymbol{F}. \tag{29}$$

This indicates that $\boldsymbol{F}$ is analogous to vorticity in fluid. The vorticity is twice the angular velocity ($\boldsymbol{\Omega}$), i.e., $\boldsymbol{\omega}_{\rm v} = 2\boldsymbol{\Omega}$, so that equation (29) reveals that the electromagnetic force acting on the particle is given by $2m\boldsymbol{\Omega}c$. This is just a Coriolis-like force. Thus, a moving charged particle will experience a deflection force of $4\pi fmc$ in an electromagnetic field (wave) of frequency, f. It seems that this is a mechanical force imparted by the fields on the charged particle, and it is interesting to see from Eqs. (26) and (29) that $\boldsymbol{\nabla} \cdot \boldsymbol{F} = 0$. Considering Eq. (2), we know that $\rho = 0$. Therefore, equation (29) is valid in free space only.

If we now write Eq. (29) as

$$\boldsymbol{\Omega} = \frac{q}{2mc}\boldsymbol{E} + \mathrm{i}\frac{q}{2m}\boldsymbol{B}, \tag{30}$$

we obtain the generalized Larmor precession.

4. Energy conservation and duality transformation

Now we multiply Eq. (2) by $\boldsymbol{F}^*$ and multiply (dot product) the complex conjugate of Eq. (2) by $\boldsymbol{F}$, and add the two resulting equations, then we obtain the energy conservation equation as

$$\frac{\partial u}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{S} = -\boldsymbol{J} \cdot \boldsymbol{N}, \tag{31}$$

where

$$u = \frac{1}{2\mu_0}|\boldsymbol{F}|^2, \ \boldsymbol{S} = \frac{\mathrm{i}c}{2\mu_0}\boldsymbol{F} \times \boldsymbol{F}^*, \ \boldsymbol{N} = \frac{c}{2}(\boldsymbol{F} + \boldsymbol{F}^*), \tag{32}$$

through employing the vector identity

$$\boldsymbol{\nabla} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}), \tag{33}$$

with $\boldsymbol{A} = \boldsymbol{F}^*$ and $\boldsymbol{B} = \boldsymbol{F}$. Note that in the ordinary Maxwell's equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{S} = -\boldsymbol{J} \cdot \boldsymbol{E}, \tag{34}$$

where

$$\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E} \times \boldsymbol{B}, \quad u = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right). \tag{35}$$

Now we pay attention to the significance of the electromagnetic field vector $\boldsymbol{F}$. To understand the nature of vector $\boldsymbol{F}$, we consider its evolution in free space (vacuum) where $\boldsymbol{J}=0$. In this case, equation (2) reads

$$\frac{\partial \boldsymbol{F}}{\partial t}=-\mathrm{i}c\boldsymbol{\nabla}\times\boldsymbol{F}. \quad (36)$$

If we write $\boldsymbol{F}$ as a plane wave of the form

$$\boldsymbol{F}=\boldsymbol{F}_0\,\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}, \quad (37)$$

then equation (37) reads

$$\frac{\partial \boldsymbol{F}}{\partial t}=\boldsymbol{\omega}\times\boldsymbol{F},\quad \boldsymbol{\omega}=c\boldsymbol{k}. \quad (38)$$

This indicates that the electromagnetic vector precesses with angular velocity $\boldsymbol{\omega}$ (here $\boldsymbol{B}$ precesses with respect to $\boldsymbol{E}$, or vice versa).

Now, we consider the duality transformation that has some internal rotations of $\boldsymbol{E}$ and $\boldsymbol{B}$ in $\boldsymbol{F}$-space by a right angle. In fact, the $\boldsymbol{F}$-space can be seen as a two-dimensional (2D) vector space spanned by $\boldsymbol{E}$ and $\boldsymbol{B}$, where

$$\boldsymbol{F}=\begin{pmatrix}\boldsymbol{E}\\ c\boldsymbol{B}\end{pmatrix}.$$

To illustrate this, we consider the rotation of the 2D space $(\boldsymbol{E},\boldsymbol{B})$ by an angle θ, namely,

$$\begin{pmatrix}\boldsymbol{E}'\\ c\boldsymbol{B}'\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\boldsymbol{E}\\ c\boldsymbol{B}\end{pmatrix},$$

where $\theta=\pi/2$. This is equivalent to the duality transformation.

Now under the duality transformation

$$\boldsymbol{E}\rightarrow c\boldsymbol{B},\quad c\boldsymbol{B}\rightarrow-\boldsymbol{E}, \quad (39)$$

equation (1) states that

$$\boldsymbol{F}\rightarrow-\mathrm{i}\boldsymbol{F}. \quad (40)$$

The invariance of Maxwell's equations (2) and (3) requires

$$\rho\rightarrow-\mathrm{i}\rho,\quad \boldsymbol{J}\rightarrow-\mathrm{i}\boldsymbol{J}. \quad (41)$$

However, equations (40) and (41) show that the energy conservation equation is not invariant under the duality transformation.

If we consider the complex space-time transformation introduced by t'Hooft and Nobbenhuis,[13] and generalized by us,[14] then it follows

$$t\rightarrow-\mathrm{i}t,\quad \boldsymbol{r}\rightarrow-\mathrm{i}\boldsymbol{r}. \quad (42)$$

Then equations (40) and (42) imply that

$$\rho\rightarrow\rho,\quad \boldsymbol{J}\rightarrow\boldsymbol{J}, \quad (43)$$

instead of the transformation in Eq. (41). It is worth mentioning that in the ordinary Maxwell's equation such a symmetry does not exist. However, the duality transformation exists for symmetric Maxwell's equations only. It is thus very intriguing that Maxwell's equations are dually invariant if they are accompanied by complex space-time transformation.

Now comparing the energy equation (32) with Dirac's equation suggests that $\boldsymbol{F}$ is tantamount to the wave function ψ, so that $|\boldsymbol{F}|^2$ resembles $|\psi|^2$, and $\boldsymbol{F}^*\times\boldsymbol{F}$ resembles the current density $\bar{\psi}\boldsymbol{\alpha}\psi$. Thus, $\boldsymbol{F}$ represents the wave function of the photon.

4.1. The CPT theorem

If we assume that Maxwell's equations are invariant under CPT (C: charge conjugation; P: parity; T: time reversal), i.e., $q\rightarrow-q$, $\boldsymbol{r}\rightarrow-\boldsymbol{r}$, and $t\rightarrow-t$. However, under time reversal, $\boldsymbol{E}(\boldsymbol{B})\rightarrow\boldsymbol{E}(-\boldsymbol{B})$ and $\boldsymbol{J}(\rho)\rightarrow-\boldsymbol{J}(\rho)$. Under parity, $\boldsymbol{E}(\boldsymbol{B})\rightarrow-\boldsymbol{E}(\boldsymbol{B})$ and $\boldsymbol{J}(\rho)\rightarrow-\boldsymbol{J}(\rho)$. Under charge conjugation, $\boldsymbol{E}(\boldsymbol{B})\rightarrow-\boldsymbol{E}(-\boldsymbol{B})$ and $\boldsymbol{J}(\rho)\rightarrow-\boldsymbol{J}(-\rho)$. Accordingly, under time reversal, $\boldsymbol{F}\rightarrow\boldsymbol{F}^*$. Under parity transformation, $\boldsymbol{F}\rightarrow-\boldsymbol{F}^*$. Under charge conjugation, $\boldsymbol{F}\rightarrow-\boldsymbol{F}$. Apparently, Maxwell's equations (2) and (3) are indeed invariant under CPT in the present complex vector formulation. Applying these transformations to Maxwell's equations (2) and (3), we find that Maxwell's equations are invariant under C, P, and T, separately.

5. Symmetric Maxwell's equations

By allowing the charge and current densities to be a complex quantity, one can define

$$\rho=\rho_{\mathrm{e}}+\mathrm{i}\frac{\rho_{\mathrm{m}}}{c\mu_0},\quad \boldsymbol{J}=\boldsymbol{J}_{\mathrm{e}}+\mathrm{i}\frac{\boldsymbol{J}_{\mathrm{m}}}{c\mu_0}, \quad (44)$$

with $\boldsymbol{J}_{\mathrm{e}}$ and $\boldsymbol{J}_{\mathrm{m}}$ defining the electric and magnetic current densities, respectively; ρ_{e} and ρ_{m} are the electric and magnetic charge densities, respectively.

The symmetric Maxwell's equations are obtained if we apply the definition (44) to Eqs. (2) and (3). This yields

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}=\frac{\rho_{\mathrm{e}}}{\varepsilon_0},\quad \boldsymbol{\nabla}\cdot\boldsymbol{B}=\rho_{\mathrm{m}}, \quad (45)$$

$$\boldsymbol{\nabla}\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t}-\boldsymbol{J}_{\mathrm{m}},\quad \boldsymbol{\nabla}\times\boldsymbol{B}=\mu_0\boldsymbol{J}_{\mathrm{e}}+\frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t}. \quad (46)$$

The invariance of Maxwell's equation under the duality transformation (39) implies that

$$\boldsymbol{J}_{\mathrm{m}}\rightarrow-c\mu_0\boldsymbol{J}_{\mathrm{e}},\ \boldsymbol{J}_{\mathrm{e}}\rightarrow\frac{\boldsymbol{J}_{\mathrm{m}}}{\mu_0 c},\ \rho_{\mathrm{m}}\rightarrow-c\mu_0\rho_{\mathrm{e}},\ \rho_{\mathrm{e}}\rightarrow\frac{\rho_{\mathrm{m}}}{c\mu_0}. \quad (47)$$

These transformations suggest that the duality transformations of magnetic and electric current densities follow those of the electric and magnetic fields, respectively.

Now using Eq. (1), Eq. (11) yields

$$\boldsymbol{f}=-cq\Big(\boldsymbol{B}-\frac{\boldsymbol{v}}{c^2}\times\boldsymbol{E}\Big)+\mathrm{i}q(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}). \quad (48)$$

Since in a rotating frame

$$\boldsymbol{B}'=\boldsymbol{B}-\frac{\boldsymbol{v}}{c^2}\times\boldsymbol{E},\quad \boldsymbol{E}'=\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B},$$

equation (48) may determine the force acting on a charge in a rotating frame with respect to the laboratory frame. Alternatively, one can determine the force term, $-c\mu_0 q(\boldsymbol{H}-\boldsymbol{v}\times\boldsymbol{D})$, where $\boldsymbol{B}=\mu_0\boldsymbol{H}$ and $\boldsymbol{D}=\varepsilon_0\boldsymbol{E}$, represent the force on a magnetic charge, $q_{\rm m}=c\mu_0 q=q/\varepsilon_0 c$. This suggests that

$$q_{\rm m}q=\frac{q^2}{\varepsilon_0 c}=2\alpha h,$$

where α is the fine structure constant, h is Planck's constant, and q is the electronic charge. However, Dirac found that $q_{\rm m}q=nh$, where n is an integer.[15] Interestingly, it was shown by Kang *et al*.[16] that the above force is due to a dyon moving in an electromagnetic field.

The Lorentz force on the electric ($q_{\rm e}$) and magnetic ($q_{\rm m}$) charges will be

$$\boldsymbol{f}_{\rm em}=q_{\rm e}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})+q_{\rm m}(\boldsymbol{H}-\boldsymbol{v}\times\boldsymbol{D}). \tag{49}$$

If we compare this force with the force in Eq. (48), we reveal that the magnetic charge is $q_{\rm m}=-c\mu_0 q_{\rm e}$. Moreover, the magnetic and electric forces are interchanged (dual). This seems to be true because in that frame the duality transformation is applied. With the aid of Eq. (49), one finds that $q_{\rm e}\to q_{\rm m}/\mu_0 c$ and $q_{\rm m}\to -c\mu_0 q_{\rm e}$. Hence, this force is the dual force of the Lorentz force in Eq. (48). Therefore, we may attribute the force in Eq. (48) to the internal force arising from the rotating magnetic field with respect to the laboratory frame. This is the dual force of the Lorentz generalized force. Moreover, the Larmor frequency will be

$$\boldsymbol{\Omega}=\frac{q_{\rm m}}{2m}\boldsymbol{D}+{\rm i}\frac{q_{\rm e}}{2m}\boldsymbol{B},$$

where the first term on the right-hand side describes the precession due to magnetic charge (magnetic current) in the presence of an electric field density, and the second one represents the precession due to electric charge (electric current) in the presence of a magnetic field. We deduce here that the magnetic charge has an electric dipole moment associated with it. This analogy deepens the fundamental relations between electricity and magnetism, and this association is supported by the recent proposal made by Khomskii[17] of a magnetic material called spin ice. This can trigger the exploration of monopoles by the application of an electric field.

The Dirac quantization condition can be seen as reflecting the impossibility of determining the electric and magnetic charges simultaneously. The charge conservation violation can be related by an uncertainty relation

$$\Delta q_{\rm e}\Delta q_{\rm m}\geq\hbar, \tag{50}$$

where $q_{\rm m}$ is related to the charge of the photon. Thus, it is impossible to accurately measure the electric charge and magnetic charge simultaneously. The magnetic charge determines the charge of the photon. Hence, Dirac's quantization determines the non-conservation of the electric and magnetic charges. Ji *et al*.[18] suggested an optical system that allows a direct experimental observation of the quantum magnetic correlated dynamics of polarized light. Moreover, an experimental possibility is studied for the case of the two-qubit Heisenberg XY model in a homogeneous magnetic field by Abliz *et al*.[19] These two kinds of studies may elucidate how photon electric and magnetic charges influence the behavior of photons.

6. Generalized Maxwell's equations

It is now pertinent to consider the generalized Maxwell's equations (2) in terms of the vector $\boldsymbol{F}$, employing quaternions.[2] To this aim, we write

$$\tilde{\nabla}^*\tilde{\boldsymbol{F}}=\mu_0\tilde{\boldsymbol{J}}, \tag{51}$$

where

$$\tilde{F}=(\Lambda,{\rm i}\boldsymbol{F}),\quad \tilde{J}=({\rm i}\rho c,\boldsymbol{J}),\quad \tilde{\nabla}^*=\Big(\frac{\rm i}{c}\frac{\partial}{\partial t},-\boldsymbol{\nabla}\Big), \tag{52}$$

and the scalar Λ defines some scalar function representing the fourth component of the electromagnetic 4-vector. The expansion of Eq. (51) by means of Eq. (52) yields

$$\frac{1}{c}\frac{\partial\Lambda}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{F}=\mu_0 c\rho, \tag{53}$$

$$-\boldsymbol{\nabla}\Lambda-\frac{1}{c}\frac{\partial\boldsymbol{F}}{\partial t}-{\rm i}\boldsymbol{\nabla}\times\boldsymbol{F}=\mu_0\boldsymbol{J}. \tag{54}$$

Equations (53) and (54) generalize Maxwell's equations to incorporate an electroscalar wave. Is it the scalar wave that was found by Tesla and called the "electric sound wave"? If so, then Tesla was also correct in his discovery!

By differentiating Eq. (53) partially with respect to time, and taking the divergence of Eq. (54), then we obtain

$$\frac{1}{c}\frac{\partial^2\Lambda}{\partial t^2}-\nabla^2\Lambda=\mu_0\Big(\boldsymbol{\nabla}\cdot\boldsymbol{J}+\frac{\partial\rho}{\partial t}\Big). \tag{55}$$

Charge conservation dictates that the right hand-side of Eq. (55) vanishes. Hence, the scalar function Λ satisfies the wave equation, and besides the electromagnetic field which has a transverse nature, a scalar field with a longitudinal character is predicted by the generalized Maxwell's equations (53) and (54). Equations (53) and (54) are invariant under duality transformations if we employ Eq. (44) and write

$$\Lambda=\Lambda_{\rm m}+\frac{\rm i}{c}\Lambda_{\rm e}, \tag{56}$$

so that under duality transformations,

$$\Lambda_{\rm e}\to -c\Lambda_{\rm m},\quad \Lambda_{\rm m}\to\frac{1}{c}\Lambda_{\rm e}. \tag{57}$$

7. The Dirac–Maxwell analogy

As clearly seen from Eq. (15), the wave nature of the electromagnetic fields arises not from the charge and current densities themselves, as generally assumed, but from their temporal and spatial distribution, and the current circulation. The charge and current density distributions are the sources for the electric and magnetic fields. Moreover, in a medium with a static charge density and steady current density, the electromagnetic field travels at the speed of light. Thus, the electromagnetic field travels at the speed of light not only in a vacuum but also in a medium with static charge and current densities. The electromagnetic field therefore arises from dynamical charge and current density systems.

We now consider the matter waves (whose wave function is ψ) devised by de Broglie and manifested by Dirac's equation [20]

$$\frac{\partial \psi}{\partial t} + c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\psi = -\frac{imc^2}{\hbar}\beta\psi, \tag{58}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are some matrices, and $\hbar = h/2\pi$is the reduced Planck's constant. This equation can be compared with Maxwell's equations (2) and (3). This comparison suggests that ψ represents the matter field of the particle. The wave nature in Eq. (58) is associated with the particle's mass (as its source), while the electromagnetic field (Eqs. (2) and (3)) has its source in the charge and current densities of the system concerned. It is interesting to note that while the electromagnetic field is related to the densities, the matter field is related to the mass of the given system.

8. Concluding remarks

In this paper we present a complex formulation of Maxwell's equations. Maxwell's equations are more compact in this formulation; they are one vector equation and one scalar equation. A new form of Maxwell's equations is found and we deduce the generalized Lorentz force from this form. Ordinary Maxwell's equations are shown to be invariant under duality transformation, if accompanied by complex space and time transformation. The new form of Maxwell's equations helps to reveal the analogy existing between the quantum mechanical equations of motion. We suggest new transformations that can lead to massive photons in Maxwell's theory. A new vacuum solution and its transformations are studied, where the electromagnetic wave travels with the speed of light in a medium with $\rho \neq 0$ and $\boldsymbol{J} \neq 0$. The relation between the electric charge and magnetic charge is shown to express the uncertainty in determining these two quantities. This relation accords with Heisenberg's uncertainty relation. The generalized Maxwell's equations are also found to be invariant under the duality transformations.

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